Notes on Transit in Deflecting Mode Pillbox Cavity

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1 Coordinate System and Reference Particle

The natural choice of reference particle is one that passes the longitudinal center of the cavity at the time when the transverse deflecting electric field is a maximum. There are a couple of choices available for transverse coordinates. Because of the magnetic fields in the cavity, the reference particle describes a "dogleg" in its passage, and one could attach the transverse coordinate center to this trajectory. But let me rather keep the longitudinal axis, s, as a straight line passing through the transverse center of the fields, and take X as the relevant transverse coordinate for the motion, anticipating that $x \equiv X - X_r$ will eventually mean motion with respect to the reference particle. Assume further that the reference particle enters on and in the direction of the s axis. The other transverse degree-of-freedom, Y plays no role other than as a label for the magnetic field direction.

The cavity fields for the TM_{110} mode near the axis of the pillbox may be written

$$\begin{aligned} E_s(X,t) &= E_0(X)\cos(\omega t) = E' X \cos(\omega t) && (1)\\ B_Y(X,t) &= B_0(X)\sin(\omega t) = \frac{E'}{\omega}\sin(\omega t) && (2)\end{aligned}$$

where the far-right expressions in each of the above follow from application of Faraday's Law and expansion to lowest non-vanishing order in X. Here $E' \equiv \partial E_X/\partial X$ evaluated at $t = 0$. For a listing of the pillbox fields without this approximation, see Sec. 6. The fields used here may be obtained from a vector portential

$$A_s = -\frac{E'X}{\omega}\sin(\omega t) \tag{3}$$

With application of $d\vec{p}/dt = e(\vec{E} + \vec{v}\times\vec{B})$, the equations of motion are

$$\begin{aligned} \frac{dP_X}{dt} &= -eE'\frac{V_s}{\omega}\sin(\omega t) && (4)\\ \frac{dP_S}{dt} &= eE'X\cos(\omega t) + eE'\frac{V_X}{\omega}\sin(\omega t) && (5)\end{aligned}$$

As a check, let's see if Eqs. 4, 5 lead to the same behavior in P_X and P_s for transit of a "thin" cavity as that used by Kim and colleagues. In π-mode, anticipating that the actual hardware will be operating in that mode, it follows that both ω and E' head toward infinity while preserving a finite ratio. With a change in variable to $\theta = \omega t$, in that limit, Eq. 4 becomes $dP_X/d\theta = 0$, so P_X does not change. Eq. 5 is now

$$\frac{dP_s}{d\theta} = \frac{eE'}{\omega} X \cos\theta - \frac{eE'V_s}{\omega^2} \sin\theta. \tag{6}$$

The second term on the right side of this equation vanishes in the high frequency limit, and integration of the first term yields

$$\frac{\Delta P_s}{P_0} = \frac{2}{\pi} \frac{eE'(\lambda/2)}{pc} X = TX, \tag{7}$$

The ratio $2/\pi$ is the transit time factor for π-mode, for which the cavity is of length $\lambda/2$. Up to the choices of sign, T is the same as the k used by other authors; I'm reserving the lower case letter for the conventional $k \equiv 2\pi/\lambda$. Explicitly

$$T \equiv \frac{2eE'}{pc} \frac{1}{k} \tag{8}$$

and I leave the matter of "which p" ambiguous for the present.

Next, return to the finite-length cavity situation, and estimate the change in transverse position. Suppose the particle enters at $X = 0$ and $dX/dt = 0$, that we can ignore $V_X d\gamma/dt$ compared to $\gamma dV_X/dt$, and that $V_s = c$. Then

$$\frac{d^2X}{d\theta^2} \approx -\frac{T}{2k^2} sin\theta \longrightarrow X = \frac{T}{k^2}\left(\frac{1+\sin\theta}{2}\right), \tag{9}$$

and at exit from the cavity

$$X = \frac{T}{k^2}. \tag{10}$$

after using the definition of T from Eq. 8. Take $T = -1/D$, where D is the value of the dispersion function, as in the single cavity models of the process. In the experiment under construction at Fermilab, $D = 0.33$ m, and $\lambda = 0.077$ m for the 3.9 GHz RF system. With these parameters, Eq. 10 gives $x = -0.44$ mm, not entirely negligible on the beam size scale for the photoinjector.

At $s = 0$, the offset is one-half of that given by Eq. 10 and $X' = T\lambda/4\pi$, so the bend center of the first element of the dogleg associated with the cavity is located upstream at $s = -\lambda/(2\pi) = -1/k$. For a single cell structure, the bend angle is 1.06 degrees for our parameters. At 15 MeV, the local coordinate system travels with speed characterized by $1 - \beta \approx 1/(2\gamma^2) = 5.78 \times 10^{-4}$ along the straight s axis, and at the angle of 1.06 degrees the reference particle is slower by a factor of $1 - \cos\theta = 1.72 \times 10^{-4}$. Looks like we're safe in taking $s = ct$ for the location of the moving coordinate system, and taking $V_s = c$ for the reference particle as well.

However, the reference particle follows the path of Eq. 9, and so we should at least calculate the change in the distance by which it leads or lags the local coordinate system origin.

$$\frac{dZ}{ds} = \frac{eB_y}{p} X = \frac{T^2}{4k^2} \left[1 - \cos(ks)\right] \sin(ks) \tag{11}$$

from which

$$Z = -\frac{T^2}{4k^3} \left[\cos(ks) - \frac{1}{4}\cos(2ks)\right] \tag{12}$$

giving a maximum value $Z(s = 0) = -7.9 \times 10^{-7}$ m. A similar calculation for path length shows that the reference particle travels about 4×10^{-8} m less than $\lambda/2$.

Finally, use X and dX/ds from Eq. 9 as input to Eq: 5 to get an estimate of the change in P_s,

$$\frac{\Delta P_s}{P_0} = \frac{1}{4}\frac{T^2}{k^2} \left[\sin(ks) - \frac{1}{2}\cos(2ks) + \frac{1}{2}\right] \tag{13}$$

with the result at exit $\Delta P_s/P_0 = 0.69 \times 10^{-3}$ which is not negligible on the scale of momentum spreads at the photoinjector.

From the last few paragraphs, the ratio T/k emerges as an expansion parameter. Observe that both X and kdX/ds have this ratio as a coefficient, while Z and $k\Delta P/P_0$ are proportional to $(T/k)^2$. Also, the scale of the excursions in all but possibly z are significant for photoinjector parameters. These calculations were done with single cell numbers; presumably the situation will be better in some respects for the 5-cell case, with T reduced by a factor of 5. Neverthless, we should certainly include the reference particle trajectory in the calculations, and probably retain second order in T/k.

2 Equations of Motion

Quantities with the subscript "r" pertain to the reference particle, and lower case means values with respect to the reference particle, *e. g.*, $x \equiv X - X_r$.

As in Sec. 1, the reference particle reaches the longitudinal center of the cavity at $t = 0$, as does a moving set of coordinate axes running along the s axis. Then a particle leading the reference particle by a distance $z \equiv s - s_r$ will reach that midpoint at $t = -z/c$. Note that $z \ll \lambda/2$; for a 10 ps laser pulse, the bunch length is 3 mm compared with the the 77/2 mm cavity length. For this particle, the fields become

$$
\begin{aligned}
E_s &= E' X \cos(\omega t - kz) \approx E' X \cos(\omega t) + E' X kz \sin(\omega t) && (14)\\
B_y &= \frac{E'}{\omega} \sin(\omega t - kz) \approx \frac{E'}{\omega} \sin(\omega t) - \frac{E'}{c} z \cos(\omega t) && (15)
\end{aligned}
$$

where E' is defined as in Eqs. 1, 2. With a change of independent variable to $s = v_s t \approx ct$ and use of $\omega = kc$, the equations of motion for the reference particle are

$$
\begin{aligned}
\frac{dP_{X,r}}{ds} &= -\frac{eE'}{kc} \sin(ks) && (16)\\
\frac{dP_{s,r}}{ds} &= eE' X_r \cos(ks) + eE' \frac{V_{X,r}}{kc} \sin(ks) && (17)
\end{aligned}
$$

Neglecting terms of second order in the dependent variables, the equations for a neighboring particle are

$$
\begin{aligned}
\frac{dP_X}{ds} &= -\frac{eE'}{kc} \sin(ks) + \frac{eE'}{c} z \cos(ks) && (18)\\
\frac{dP_s}{ds} &= \frac{eE'}{c} X \cos(ks) + \frac{eE'}{kc^2} V_x \sin(ks) && (19)
\end{aligned}
$$

The difference of the preceding sets of equations yields

$$
\begin{aligned}
\frac{dp_x}{ds} &= \frac{eE'}{c} z \cos(ks) && (20)\\
\frac{dp_s}{ds} &= \frac{eE'}{c} x \cos(ks) + \frac{eE'}{kc} \frac{dx}{ds} \sin(ks) && (21)
\end{aligned}
$$

The equations of motion are completed with

$$
\frac{dx}{ds} = \frac{p_x}{p_1}; \quad \frac{dz}{ds} = \frac{eB_Y}{p_1} X \approx \frac{eE'}{p_1 ck} x \sin(ks) \tag{22}
$$

where p_1 is the longitudinal momentum at entry to the cavity.

Integration of Eq. 20 gives

$$p_x(s) = p_x(-\lambda/4) + \frac{eE'}{ck} z \left[\sin(ks) + 1\right] \tag{23}$$

With the definition of x'

$$x' \equiv \frac{p_x}{p_s} = \frac{\gamma m v_x}{\gamma m v_s} = \frac{1}{v_s}\frac{dx}{dt} = \frac{dx}{ds} \tag{24}$$

Eq. 23 becomes

$$x'(s) = x'_1 + \frac{1}{2} T z \sin(ks) + \frac{Tz}{2} \tag{25}$$

where x'_1 is the value at entry to the cavity. Integration of this result gives

$$x(s) = x_1 + \left(x'_1 + \frac{T}{2} z\right)\left(s + \frac{\lambda}{4}\right) - \frac{T}{2k} z \cos(ks) \tag{26}$$

Eq. 21 can now be integrated after insertion of the results from Eqs. 25, 26 with the result

$$\frac{\delta p}{p} \equiv \frac{p_s(s) - p_0}{p_0} = \left(\frac{\delta p}{p}\right)_1 + x_1 \frac{T}{2}\left[1 + \sin(ks)\right] \tag{27}$$

$$+ \quad x'_1 \frac{T}{2}\left[\left(s + \frac{\lambda}{4}\right)\sin(ks)\right] \tag{28}$$

$$+ \quad z\frac{T^2}{4}\left[\left(s + \frac{\lambda}{4}\right)\sin(ks) + \frac{1}{k}\cos(ks) - \frac{1}{2k}\sin(2ks)\right] \tag{29}$$

where p_0 is a reference momentum and inhomogeneous terms have been omitted.

3 Application to Phase Space Exchange

The version of the transverse-longitudinal interchange process invented by Kwang-Je Kim uses a dogleg-cavity-dogleg sequence. In the thin bend magnet representation, the dogleg matrix is given by

$$M_d = \begin{pmatrix} 1 & L & 0 & \alpha L \\ 0 & 1 & 0 & 0 \\ 0 & \alpha L & 1 & \alpha^2 L \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & D/\alpha & 0 & D \\ 0 & 1 & 0 & 0 \\ 0 & D & 1 & \alpha D \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{30}$$

where α is the magnitude of the bending angle in each magnet, L the distance between the two bends of a dogleg, the dogleg offset is in the negative x direction, and the right-most form of the matrix is written in terms of the dispersion $D = \alpha L$.

Upstream of the exchange apparatus, assume that the beam line is non-dispersive, as in the Fermilab photoinjector, the input and output states of the first dogleg are related by

$$\begin{pmatrix} x_1 \\ x_1' \\ z_1 \\ (\delta p/p)_1 \end{pmatrix} = \begin{pmatrix} x_0 + Dx_0'/\alpha + D\,(\delta p/p)_0 \\ x_0' \\ z_0 + Dx_0' + \alpha D\,(\delta p/p)_0 \\ (\delta p/p)_0 \end{pmatrix} \tag{31}$$

where it may be noted that z is not constant through a dogleg in contrast to the assumption about the cavity. From the results of Sec. 2, we can write passage through a cavity as

$$\begin{pmatrix} x_2 \\ x_2' \\ z_2 \\ (\delta p/p)_2 \end{pmatrix} = \begin{pmatrix} x_1 + \frac{\lambda}{2}x_1' + \frac{\lambda}{4}Tz \\ x_1' + Tz \\ z_1 \\ (\delta p/p)_1 + Tx_1 + \frac{T\lambda}{4}x_1' + \frac{T^2\lambda}{8} \end{pmatrix} \tag{32}$$

or equivalently write a cavity matrix as

$$M_c = \begin{pmatrix} 1 & \lambda/2 & T\lambda/4 & 0 \\ 0 & 1 & T & 0 \\ 0 & 0 & 1 & 0 \\ T & T\lambda/4 & T^2\lambda/8 & 1 \end{pmatrix} \tag{33}$$

Now one wonders if the finite-length cell preserves the perfect phase space interchange potential. It does at the expense of ignoring the 4,3 element of the cavity. With $T = -1/D$ and setting the 4,3 element of M_c to zero, the dogleg-cavity-dogleg product gives

$$M_d M_c M_d = \begin{pmatrix} 0 & 0 & -\frac{1}{\alpha} - \frac{\lambda}{4D} & -\frac{\alpha\lambda}{4} \\ 0 & 0 & -\frac{1}{D} & -\alpha \\ -\alpha & -\frac{\alpha\lambda}{4} & 0 & 0 \\ -\frac{1}{D} & -\frac{1}{\alpha} - \frac{\lambda}{4D} & 0 & 0 \end{pmatrix} \tag{34}$$

Eq. 34 is of the same exchange-form as the thin cavity limit, with the addition of λ/D terms to the off-diagonal 2×2 matrices, but at the cost of ignoring one matrix element. With the inclusion of the 4,3 element from Eq. 33 the result equivalent to Eq. 34 is

$$M_d M_c M_d = \begin{pmatrix} 0 & \lambda/8 & -\frac{1}{\alpha} - \frac{\lambda}{8D} & -\frac{\alpha\lambda}{8} \\ 0 & 0 & -\frac{1}{D} & -\alpha \\ -\alpha & -\frac{\alpha\lambda}{8} & \alpha\lambda/(8D) & \alpha^2\lambda/8 \\ -\frac{1}{D} & -\frac{1}{\alpha} - \frac{\lambda}{8D} & \lambda/(8D^2) & \alpha\lambda/(8D) \end{pmatrix} \tag{35}$$

and the diagonal 2×2 submatrices are becoming populated. The symplectic condition is no help; the matrix of Eq. 33 is symplectic regardless of the 4, 3 element.

For the 5-cell case at Fermilab, we take $T = -1/(5D)$ and the matrix becomes

$$M_d S_2 M_c^5 S_1 M_d = \begin{pmatrix} 0 & \frac{17\lambda}{40} & -\frac{1}{\alpha} - \frac{33\lambda}{40D} - \frac{s_2}{D} & -\frac{33\alpha\lambda}{40} - \alpha s_2 \\ 0 & 0 & -\frac{1}{D} & -\alpha \\ -\alpha & -\frac{33\alpha\lambda}{40} - \alpha s_1 & \frac{17\alpha\lambda}{40D} & \frac{17\alpha^2\lambda}{40} \\ -\frac{1}{D} & -\frac{1}{\alpha} - \frac{33\lambda}{40D} - \frac{s_1}{D} & \frac{17\lambda}{40D^2} & \frac{17\alpha\lambda}{40D} \end{pmatrix} \tag{36}$$

where drifts of length s_1 and s_2 have been inserted upstream and downstream of the 5-cell respectively. These drift-related terms only appear in the off-diagonal 2×2's and so do not detract from the phase space interchange behavior. Numerical calculation gives

$$M_d S_2 M_c^5 S_1 M_d = \begin{pmatrix} 0.000 & 0.0327 & -4.679 & -0.276 \\ 0.000 & 0.000 & -3.030 & -0.393 \\ -0.393 & -0.276 & 0.039 & 0.005 \\ -3.030 & -4.679 & 0.301 & 0.039 \end{pmatrix}. \tag{37}$$

where the entries are in mks units. The drifts on either side of the cavity have been taken to be 0.64 m each. The 4,3 matrix element continues to grow with cell number. Since z for a few picosecond laser pulse z can be of the order of a millimeter and $\delta p/p$ can be in the 10^{-3} range, this could be a significant perturbation. In principle, this effect could be measured through variation of bunch length or timing.

4 Different Approach as a Check

Let's try a fixed x, y, z coordinate system, with the origin at the upstream end of the cavity. and the z-axis directed along the cavity center line. The Hamiltonian may be written

$$H = \left[p_x^2 c^2 + (\mathfrak{P}_z - eA_z)^2 c^2 + m^2 c^4\right]^{1/2} \tag{38}$$

where $\mathfrak{P}_z$ denotes the z-component of the canonical momentum. No distinction need be made between the canonical and kinematic momenta in the x-direction. The vector potential is

$$A_z = \frac{E'}{\omega} x \cos(\omega t - \phi) \tag{39}$$

where $t = 0$ at entry to the cavity and ϕ adjusts the entry phase. It would be expected that the reference particle of Sec. 1 would enter at very close to $\phi = 0$.

Since z does not appear in the Hamiltonian, $\mathfrak{P}_\mathfrak{z}$ is a constant of the motion. In terms of the kinematic momentum p_z

$$p_z = p_1 - \frac{eE'}{\omega}\left[x \cos(\omega t - \phi) - x_1 \cos\phi\right] \tag{40}$$

where p_1 and x_1 are the values at $t = 0$. Let p_0 be the momentum associated with on-axis entry to the cavity. Then with the definitions

$$\delta_1 \equiv \frac{p_1 - p_0}{p_0}, \quad \delta_z \equiv \frac{p_z - p_0}{p_0} \tag{41}$$

Eq. 40 can be written

$$\delta_z = \delta_1 - \frac{eE'}{p_0 ck}\left[x \cos(\omega t - \phi) - x_1 \cos\phi\right] \tag{42}$$

This last exhibits the same behavior in the high frequency limit as a similar relation in Sec. 1. Suppose $x_1 = D\delta_1$, at the output $x_2 = x_1$, $\omega t = \pi/2$, $\phi = 0$, then with $T \equiv 2eE'/p_0 ck = -1/D$, the result is $\delta_2 = 0$ just as in the earlier treatment.

For p_x, we have

$$\begin{aligned} \frac{dp_x}{dt} = -\frac{\partial H}{\partial x} &= \frac{eE'}{k}\frac{p_z}{\gamma mc}\cos(\omega t - \phi) && (43) \\ &= \frac{eE'}{k}\beta_z \cos(\omega t - \phi) && (44) \end{aligned}$$

The deviation of β_z from unity is mostly due to dx/ds. If that is as large as a milliradian, then $1 - \beta_z \approx 1/(2x'^2) \approx 10^{-6}$. That may be large enough to try to include. For the moment, take $\beta_z = 1$ to compare with earlier results. Then integration of Eq. 44 gives

$$x'(z) = x'_1 + \frac{T}{2k}\sin(kz-\phi) + \frac{T}{2k}\sin\phi \tag{45}$$

$$x(z) = x_1 + x'_1 z - \frac{T}{2k^2}\cos(kz-\phi) + \frac{T}{2k^2}\cos\phi + \frac{T}{2k} z\sin\phi \tag{46}$$

where I have made the substitution $\omega t = kz$, and this z is not the distance between the particle of interest and the reference particle. Putting these expressions into Eq. 42 gives at $z = \lambda/2$:

$$\delta_2 = \delta_1 + Tx_1 + \frac{T\lambda}{4}x'_1 + \frac{T^2\lambda}{8}\frac{\sin 2\phi}{2k} + \frac{T^2}{2k^2}. \tag{47}$$

But if we revert to the circumstance where the z coordinate is defined with respect to the reference particle, then we would set $\phi = kz \ll 1$ and these results would be identical with those of Sec. 2 as they are.

5 Some Derivatives

5.1 Cavity Length

Suppose that the cavity (or cell) is longer than $\lambda/2$ in length by δL, but that the frequency is correct. Assume further that the reference particle will still cross cavity center at $t = 0$ as in Sec. 1 and Sec. 2. Instead of a phase difference of $\pi/2$ between entrance and midpoint, the phase difference is $\pi/2 + \pi\delta L/\lambda$, with the same excess on the way out. Integration of Eq. 9 gives at output

$$\frac{dX}{d\theta} = \frac{T}{k^2}\frac{\pi\delta L}{2\lambda} \tag{48}$$

$$X = \frac{T}{k^2}\left(1 + \frac{\pi^2\delta L}{2\lambda}\right) \tag{49}$$

For $\delta L = 1.5$ mm, the result is a 20% increase in the offset of the reference particle trajectory.

6 Relation to Standard Pillbox Fields

The field equations of Sec. 1 were written in the paraxial approximation as is usual for linear dynamics. For the "real" pillbox, the TM-110 mode fields as obtained from page 41 of Padamsee, Knobloch and Hays are

$$E_s = \tilde{E}_0 J_1\left(\frac{u_1 r}{R}\right)\cos\phi\cos(\omega t) \tag{50}$$

$$E_r = 0, \quad E_\phi = 0, \quad H_s = 0 \tag{51}$$

$$H_r = \tilde{E}_0\frac{\omega R^2}{Z_0 c u_1^2}\frac{1}{r}J_1\left(\frac{u_1 r}{R}\right)\sin\phi\sin(\omega t) \tag{52}$$

$$H_\phi = \tilde{E}_0\frac{\omega R}{Z_0 c u_1}J_1'\left(\frac{u_1 r}{R}\right)\cos\phi\sin(\omega t) \tag{53}$$

$$\omega = \frac{c u_1}{R} \tag{54}$$

where an accent has been place on E_0 for here it is a constant in contrast with the $E_0(x)$ of Sec. 1, $u_1 = 3.832$ is the first root of $J_1(x) = 0$ for $x \neq 0$, R is the pillbox radius, and $Z_0 = 377$ohms is the impedance of free space.

The implication of the earlier text is that $r \approx 1$ mm is the transverse scale of interest. According to Eq. 54, $R = 29$ mm at resonance. Therefore activities at a millimeter or so from the axis are within the linear regime. Of likely more interest are the deviations from the simple pillbox model resulting from the actual resonator structure.